A THEORY OF LINEARIZED EXPLOSION PROBLEMS INCLUDING BACK PRESSURE

(K TEORII LINEARIZIROVANNYKH ZADACH O VZRYVE C UCHETOM PROTIVODAVLENIIA)

PMM Vol.23, No.4, 1959, pp. 749-759

V.P. KOROBEINIKOV and E.V. RIAZANOV
(Moscow)

(Received 7 February 1959)

The statement and complete analytical solution of the one-dimensional self-similar problem of a strong point explosion in a gas is given by Sedov [1] (see also [2]). Here it was assumed that the initial density of the gas ρ, either is constant or depends on the geometrical coordinate r according to the following law:

$$\rho_1(r) = Ar^{-\omega}, \quad \omega, \quad A - \text{const}, \ A > 0 \qquad (0.1)$$

In addition, it was considered that the initial pressure p_1 of the undisturbed gas can be neglected in comparison with the pressure p_2 at the shock wave front. It is known that the latter assumption is correct only for the initial stage of the explosion development, i.e. for a brief time interval. As the shock wave propagates the influence of the initial pressure becomes essential. Therefore, in the initial conditions of the problem there appears an additional dimensional parameter p_1, by virtue of which the problem ceases to be self-similar and all non-dimensional characteristics of the flow will now depend not on one but on two variables. The non self-similar problem can be solved by numerical integration of a system of nonlinear equations, and this has been done for the case of constant initial density [3] with the help of high-speed computing machines.

However, a simpler means of solving the problem for values of $(p_2 - p_1)/p_1 < 1$ can be proposed, which is based on a linearization of the basic equations about the self-similar solution. For the problem of an explosion in a perfect gas with constant density and adiabatic exponent $\gamma = 1.4$ linearized solutions have been obtained previously [1,4,5]. In those papers the linearized non-dimensional parameter is q, which characterizes the intensity of the shock wave and is equal to the ratio of the square of the velocity of sound in the undisturbed gas to the square of the velocity of the shock wave:

$$q = a_1^2 c^{-2} = \gamma p_1 \rho_1^{-1} c^{-2} \tag{0.2}$$

In the present paper a solution of the linearized problem of an explosion taking into account back pressure in a medium with varying initial density, which is determined by formula (0.1), will be considered. Using the method of linearization on the variable parameter q, equations which describe one-dimensional motions similar to the self-similar cases are derived. A first integral of the obtained system of linearized equations is found and an exact analytical solution of the problem is given for

$$\omega = \omega_1 = \frac{3\nu - 2 + \gamma(2 - \nu)}{\gamma + 1}, \tag{0.3}$$

where $\nu = 3$ for spherical waves, $\nu = 2$ for cylindrical waves, and $\nu = 1$ for plane waves.

1. Statement of the problem and the basic equations. We shall take the initial system of equations of gas dynamics, which describe the one-dimensional adiabatic disturbed motions of a perfect gas behind the shock wave front, in the form

$$\begin{aligned} &\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \\ &\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial r} + \rho\left(\frac{\partial v}{\partial r} + \frac{\nu - 1}{r}v\right) = 0 \\ &\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial r} + \gamma p\left(\frac{\partial v}{\partial r} + \frac{\nu - 1}{r}v\right) = 0 \end{aligned} \tag{1.1}$$

For the solution of the problem of a point explosion it is required to find a solution of system (1.1) with the boundary conditions at the shock wave front:

$$v_2 = \frac{2}{\gamma + 1}(1 - q)c, \qquad \rho_2 = \frac{\gamma + 1}{\gamma - 1 + 2q}\rho_1, \qquad p_2 = \frac{2\gamma - (\gamma - 1)q}{(\gamma + 1)q}p_1 \tag{1.2}$$

Quantities immediately behind the shock wave front are denoted by the index 2. Let $r_2(t)$ be the radius of the shock wave. Then

$$v_2 = v(r_2, t), \quad \rho_2 = \rho(r_2, t), \quad p_2 = p(r_2, t), \quad c = dr_2/dt$$

The dependence of $v_2(t)$, $\rho_2(t)$, $p_2(t)$, $r_2(t)$ on time is unknown beforehand, their determination being equivalent to the determination of the dependence of $q(t)$. In addition to conditions (1.2), we also have the boundary condition for the velocity at the center of symmetry

$$v(0, t) = 0 \tag{1.3}$$

At time $t = 0$ a finite energy E_0 is released at the center of symmetry and the initial conditions

$$v(r, 0) = 0, \quad \rho(r, 0) = \rho_1(r) = Ar^{-\omega}, \quad p(r, 0) = p_1 = \text{const}, \quad r_2(0) = 0 \tag{1.4}$$

are given.

From the system which defines the parameters of this problem A, p_1, E_0, γ, ω, r, t it follows that the required non-dimensional functions

$$f = v / c, \qquad g = \rho / \rho_2, \qquad h = p / p_2 \tag{1.5}$$

will depend on two non-dimensional variables, for which we take

$$\lambda = r / r_2, \qquad q = a_1^2 / c^2$$

and on the constant parameters γ and ω.

Passing to the introduced non-dimensional variables and taking into account that

$$\frac{\partial}{\partial r} = \frac{1}{r_2}\frac{\partial}{\partial \lambda}, \qquad \frac{\partial}{\partial t} = \frac{c}{r_2}\left(r_2\frac{dq}{dr_2}\frac{\partial}{\partial q} - \lambda\frac{\partial}{\partial \lambda}\right)$$

$$\frac{2}{c^2}\frac{dc}{dt} = \frac{2}{c}\frac{dc}{dr_2} = \frac{d\ln c^2}{dr_2} = \left(\omega - \frac{r_2}{q}\frac{dq}{dr_2}\right)r_2$$

$$\frac{1}{c}\frac{d\ln\rho_2}{dt} = \frac{d\ln\rho_2}{dr_2} = -\frac{1}{r_2}\left(\omega + \frac{2}{\gamma - 1 + 2q}r_2\frac{dq}{dr_2}\right)$$

$$\frac{1}{c}\frac{d\ln p_2}{dt} = \frac{d\ln p_2}{dr_2} = -\frac{2\gamma}{[2\gamma - (\gamma - 1)q]q}\frac{dq}{dr_2}$$

system (1.1) can be transformed into the form

$$\begin{aligned}
&(f - \lambda)\frac{df}{d\lambda} + \frac{(\gamma - 1 + 2q)[2\gamma - (\gamma - 1)q]}{\gamma(\gamma + 1)^2}\frac{1}{g}\frac{\partial h}{\partial \lambda} + \left(\frac{\partial f}{\partial q} - \frac{f}{2q}\right)r_2\frac{dq}{dr_2} + \frac{\omega}{2}f = 0\\
&(f - \lambda)\frac{\partial\ln g}{\partial\lambda} + \frac{\partial f}{\partial\lambda} + \frac{\nu - 1}{\lambda}f + \left(\frac{\partial\ln g}{\partial q} - \frac{2}{\gamma - 1 + 2q}\right)r_2\frac{dq}{dr_2} - \omega = 0\\
&(f - \lambda)\frac{\partial\ln h}{\partial\lambda} + \gamma\left(\frac{\partial f}{\partial\lambda} + \frac{\nu - 1}{\lambda}f\right) + \left(\frac{\partial\ln h}{\partial q} - \frac{2\gamma}{[2\gamma - (\gamma - 1)q]q}\right)r_2\frac{dq}{dr_2} = 0
\end{aligned} \tag{1.6}$$

We shall define the non-dimensional radius R_2 of the shock wave by the formula

$$R_2 = r_2 / r^\circ \qquad \left(r^\circ = (E_0 / p_1)^{\frac{1}{\nu}}\right)$$

where r^0 is a characteristic dynamic length.

In order to obtain the complete solution of the problem in the adopted variables, it is necessary to determine $f(\lambda, q)$, $g(\lambda, q)$, $h(\lambda, q)$ and also of $R_2(q)$. For this it is necessary to find a solution of system (1.6) in the λ, q plane inside the square $0 \leqslant \lambda \leqslant 1$ and $0 < q < 1$ which satisfies the following boundary conditions:

at the shock

$$f(1, q) = g(1, q) = h(1, q) = 1 \qquad \text{for } \lambda = 1 \tag{1.7}$$

at the center of symmetry

$$f(0, q) = 0 \qquad \text{for } \lambda = 0 \tag{1.8}$$

and the initial conditions

$$f(\lambda, 0) = f_0(\lambda), \qquad g(\lambda, 0) = g_0(\lambda), \qquad h(\lambda, 0) = h_0(\lambda) \qquad \text{for } q = 0 \tag{1.9}$$

where $f_0(\lambda)$, $g_0(\lambda)$, $h_0(\lambda)$ are known functions corresponding to the self-similar problem [1,2]. They satisfy the system

$$\begin{gathered}(f_0 - \lambda) f_0' + \frac{2(\gamma - 1)}{(\gamma + 1)^2}\frac{h_0'}{g_0} + \frac{\omega - \nu}{2} f_0 = 0 \\ (f_0 - \lambda)\frac{g_0'}{g_0} + f_0' + \frac{\nu - 1}{\lambda} f_0 - \omega = 0 \\ (f_0 - \lambda)\frac{h_0'}{h_0} + \gamma\left(f_0' + \frac{\nu - 1}{\lambda} f_0\right) - \nu = 0\end{gathered} \tag{1.10}$$

in addition

$$f_0(1) = \frac{2}{\gamma + 1}, \qquad g_0(1) = h_0(1) = 1, \qquad f_0(0) = 0 \tag{1.11}$$

Here, as also in all that follows, primes will denote differentiation with respect to λ. We note that system (1.10) can be obtained from (1.6) by passing to the limit as $q \to 0$.

For small values of q, i.e. for small values of the time, when the explosion is still sufficiently strong, the solution of the problem set down above can be sought in the form

$$\begin{aligned} f(\lambda, q) &= f_0(\lambda) + q f_1(\lambda) + \ldots \\ g(\lambda, q) &= g_0(\lambda) + q g_1(\lambda) + \ldots \qquad \frac{R_2}{q}\frac{dq}{dR_2} = \frac{\nu}{1 + A_1 q + \ldots} \\ h(\lambda, q) &= h_0(\lambda) + q h_1(\lambda) + \ldots \end{aligned} \tag{1.12}$$

Because a linearized problem is to be solved, then from (1.6), taking into account system (1.10) and neglecting terms of order q^2 and higher, we obtain the following system of linear equations for determining the functions $f_1(\lambda)$, $g_1(\lambda)$, $h_1(\lambda)$ and the constant A_1:

$$\begin{aligned} &(f_0 - \lambda) g_0 f_1' + \frac{2(\gamma - 1)}{(\gamma + 1)^2} h_1' + \left(f_0' + \frac{\omega + \nu}{2}\right) g_0 f_1 + \\ &+ \left[(f_0 - \lambda) f_0' + \frac{\omega - \nu}{2} f_0\right] g_1 + \frac{4\gamma - (\gamma - 1)^2}{\gamma(\gamma + 1)^2} h_0' + \frac{\nu}{2} f_0 g_0 A_1 = 0 \\ &g_0 f_1' + (f_0 - \lambda) g_1' + \left(\frac{\nu - 1}{\lambda} g_0 + g_0'\right) f_1 + \end{aligned} \tag{1.13}$$

$$+\left(\frac{\nu-1}{\lambda}f_0+f_0'+\nu-\omega\right)g_1-\frac{2\nu}{\gamma-1}g_0=0$$

$$\gamma h_0f_1'+(f_0-\lambda)h_1'+\left(h_0'+\gamma\frac{\nu-1}{\lambda}h_0\right)f_1+\gamma\left(f_0'+\frac{\nu-1}{\lambda}f_0\right)h_1-$$
$$-\nu\left(\frac{\gamma-1}{2\gamma}-A_1\right)h_0=0$$

From conditions (1.7), (1.8), taking into account (1.11) and (1.12), we obtain boundary conditions for the sought for functions $f_1(\lambda)$, $g_1(\lambda)$ and $h_1(\lambda)$:

$$f_1(1)=-\frac{2}{\gamma+1},\qquad g_1(1)=h_1(1)=f_1(0)=0 \tag{1.14}$$

System (1.13) can be transformed into a form that is more convenient for the investigations which follow. With this aim we shall introduce new unknown functions $F(\lambda)$, $G(\lambda)$ and $H(\lambda)$ which are connected to the functions $f_1(\lambda)$, $g_1(\lambda)$ and $h_1(\lambda)$ by the relations

$$f_1(\lambda)=(f_0-\lambda)F,\qquad g_1(\lambda)=g_0G,\qquad h_1(\lambda)=h_0H \tag{1.15}$$

After the transformation, system (1.3) is written thus:

$$(f_0-\lambda)^2F'+\frac{2(\gamma-1)}{(\gamma+1)^2}\frac{h_0'}{g_0}H'+(f_0-\lambda)\left(2f_0'+\frac{\omega+\nu-2}{2}\right)F+ \tag{1.16}$$
$$+\frac{2(\gamma-1)}{(\gamma+1)^2}\frac{h_0'}{g_0}H+\left[(f_0-\lambda)f_0'+\frac{\omega-\nu}{2}f_0\right]G+\frac{4\gamma-(\gamma-1)^2}{\gamma(\gamma+1)^2}\frac{h_0'}{g_0}+\frac{\nu}{2}f_0A_1=0$$

$$(f_0-\lambda)F'+(f_0-\lambda)G'+(f_0'-1)F+\left(\frac{\nu-1}{\lambda}+\frac{g_0'}{g_0}\right)(f_0-\lambda)F+$$
$$+\left(\frac{\nu-1}{\lambda}f_0+f_0'+\nu-\omega\right)G+(f_0-\lambda)\frac{g_0'}{g_0}G-\frac{2\nu}{\gamma-1}=0 \tag{1.17}$$

$$\gamma(f_0-\lambda)F'+(f_0-\lambda)H'+(f_0'-1)\gamma F+(f_0-\lambda)\left(\frac{h_0'}{h_0}+\gamma\frac{\nu-1}{\lambda}\right)F+$$
$$+(f_0-\lambda)\frac{h_0'}{h_0}H+\gamma\left(f_0'+\frac{\nu-1}{\lambda}f_0\right)H-\nu\left(\frac{\gamma-1}{2\gamma}-A_1\right)=0 \tag{1.18}$$

From the two latter equations of this system it is possible to obtain a first integral. Let us show this.

2. An integral of system (1.16)-(1.18) and a law of shock wave motion. If the quantities g_0'/g_0 and h_0'/h_0 be eliminated from equations (1.17) and (1.18), then we will have

$$(f_0-\lambda)(F'+G')+(\omega-\nu)F+\nu G-\frac{2\nu}{\gamma-1}=0$$
$$(f_0-\lambda)(\gamma F'+H')-\nu(\gamma-1)F+\nu H-\nu\left(\frac{\gamma-1}{2\gamma}-A_1\right)=0$$

Replacing the independent variable according to the formula

$$\mu = \int_{\lambda}^{1} \frac{d\lambda}{f_0 - \lambda}$$

we obtain

$$\frac{d}{d\mu}(F + G) + (\omega - \nu) F + \nu G - \frac{2\nu}{\gamma - 1} = 0$$

$$\frac{d}{d\mu}(\gamma F + H) - \nu(\gamma - 1) F + \nu H - \nu\left(\frac{\gamma - 1}{2\gamma} - A_1\right) = 0$$

Multiplying now the first equation by $\nu(2\gamma - 1)/(\omega - 2\nu)$ adding to the relation so obtained the second equation and integrating the result, we find

$$\left[\frac{\nu(2\gamma - 1)}{\omega - 2\nu} + \gamma\right] F + \frac{\nu(2\gamma - 1)}{\omega - 2\nu} G + H = C_1 \exp(-\nu\mu) + \frac{2\nu}{\gamma - 1}\frac{2\gamma - 1}{\omega - 2\nu} + \frac{\gamma - 1}{2\gamma} - A_1$$

From the boundary conditions

$$F(1) = \frac{2}{\gamma - 1}, \qquad G(1) = 0, \qquad H(1) = 0, \qquad \mu(1) = 0$$

we find the constant of integration

$$C_1 = \frac{(3\gamma - 1)(\gamma + 1)}{2\gamma(\gamma - 1)} + A_1$$

If we make use of the adiabatic equation for the self-similar motions

$$(f_0 - \lambda)\left(\frac{h_0'}{h_0} - \gamma \frac{g_0'}{g_0}\right) + \gamma\omega - \nu = 0$$

and if we take into account that $\mu = 0$, $g_0 = 1$, $h_0 = 1$ for $\lambda = 1$, then we obtain

$$e^{-\mu} = (h_0 / g_0^{\gamma})^{\frac{1}{\gamma\omega - \nu}}$$

Thus, a first integral of system (1.16)-(1.18) is found which satisfies the boundary conditions at the shock wave:

$$\left[\frac{\nu(2\gamma - 1)}{\omega - 2\nu} + \gamma\right] F + \frac{\nu(2\gamma - 1)}{\omega - 2\nu} G + H =$$

$$= \left[\frac{(3\gamma - 1)(\gamma + 1)}{2\gamma(\gamma - 1)} A_1\right]\left(\frac{h_0}{g_0^{\gamma}}\right)^{\frac{\nu}{\gamma\omega - \nu}} + \frac{2\nu}{\gamma - 1}\frac{2\gamma - 1}{\omega - 2\nu} + \frac{\gamma - 1}{2\gamma} - A_1 \qquad (2.1)$$

The existence of an integral analogous to (2.1) was proved by Lidov [6].

With the help of the obtained integral (2.1) the problem reduces to the solution of a system of two linear equations, which for arbitrary values of ω can be found by numerical integration.

After finding the quantities A_1 the dependence of $R_2(q)$ and $\tau(q)$ can be found, where $R_2 = r_2/r^0$ and $\tau = t/t^0$. Here r^0 is a dynamic length, introduced earlier, and t^0 is a dynamic time defined by the formula

$$t^\circ = E_0^{1/\nu} A^{1/2} \; r^{\circ - \omega/2} p_1^{-(\nu+2)/2\nu}$$

It is known [1] that for the self-similar solution we have

$$r_2(t) = \left(\frac{E_0}{\alpha A}\right)^{\delta/2} t^\delta, \qquad c^2 = \delta^2 r_2{}^2 t^{-2}, \qquad \delta = \frac{2}{\nu + 2 - \omega} \tag{2.2}$$

where $\alpha(\gamma, \omega)$ is a known quantity.

If equations (2.2) be transformed to the introduced non-dimensional parameters, then the dependence of $R_2(q)$ and $\tau(q)$ can be found in the self-similar problem:

$$R_2^\nu(q) = \frac{\delta^2}{\gamma\alpha} q, \qquad \tau(q) = \left(\frac{\delta^2}{\gamma\alpha}\right)^{\frac{2-\nu\delta}{2\nu\delta}} \left(\nu\gamma^{1/2}\right)^{-1} q^{\frac{1}{\nu\delta}} \tag{2.3}$$

For the linearized problem of (1.12), by integrating and taking into account (2.3), we obtain

$$R_2^\nu(q) = \frac{\delta^2}{\gamma\alpha} q \exp(A_1 q) \tag{2.4}$$

Let us find $\tau(q)$. Using the definitions of q, τ, R_2, it is easy to show that

$$\frac{dR_2}{d\tau} = \left(\frac{\gamma R_2^\omega}{q}\right)^{1/2}$$

Since $d\tau/dq = (dR_2/dq)(d\tau/dR_2)$, taking into account (2.4) we find

$$\frac{d\tau}{dq} = \left(\frac{\delta^2}{\gamma\alpha}\right)^{\frac{2-\omega}{2\nu}} \left(\nu\gamma^{1/2}\right)^{-1} (1 + A_1 q)\, q^{\frac{2-\nu-\omega}{2\nu}} \exp\left(\frac{2-\omega}{2\nu} A_1 q\right)$$

For small values of q it is possible to write

$$(1 + A_1 q) \exp\left(\frac{2-\omega}{2\nu} A_1 q\right) = 1 + \frac{2\nu + 2 - \omega}{2\nu} A_1 q$$

Thus, for the determination of $\tau(q)$ we obtain the differential equation

$$\frac{d\tau}{dq} = \left(\frac{\delta^2}{\gamma\alpha}\right)^{\frac{2-\omega}{2\nu}} \left(\nu\gamma^{1/2}\right)^{-1} \left[1 + \frac{2\nu + 2 - \omega}{2\nu} A_1 q\right] q^{\frac{2-\nu-\omega}{2\nu}}$$

Integrating this equation and determining the constant of integration from the condition $\tau(0) = 0$, we find the required dependence

$$\tau(q) = \left(\frac{\delta^2}{\gamma\alpha}\right)^{\frac{2-\nu\delta}{2\nu\delta}} \left(\nu\gamma^{1/2}\right)^{-1} q^{\frac{1}{\nu\delta}} \left[\nu\delta + \frac{\nu\delta + 2}{2(\nu\delta + 1)} A_1 q\right] \tag{2.5}$$

Relations (2.4) and (2.5) give in parametric form the law of the shock wave motion, i.e. the dependence of $R_2(\tau)$.

Using (2.4) and (2.5) and the conditions at the shock wave (1.2), it is possible to determine the dependence of all characteristics of the shock wave front from its radius and the time.

3. Exact solution of the problem for $\omega = \omega_1$. It was shown earlier that the solution of the linearized problem of an explosion in a medium with variable density can always be obtained by numerical integration of system (1.16)-(1.18). However, in the case for which

$$\omega = \omega_1 = \frac{3\nu - 2 + \gamma(2-\nu)}{\gamma+1}$$

the solution of this problem can be given in the form of closed formulas. This can be explained since for this value of ω the self-similar solution has the simple form

$$f_0(\lambda) = \frac{2}{\gamma+1}\lambda, \qquad g_0(\lambda) = \lambda^{\nu-2}, \qquad h_0(\lambda) = \lambda^{\nu} \tag{3.1}$$

Substituting $f_0(\lambda)$, $g_0(\lambda)$, $h_0(\lambda)$ from (3.1) into the coefficients of equations (1.16)-(1.18), we find a system of three ordinary non-homogeneous equations with coefficients which depend on the parameters γ and ν.

$$\begin{aligned} &\lambda F' + \frac{2}{\gamma-1}\lambda H' - \frac{2(\nu+1)}{\gamma-1}F - \frac{2\nu}{\gamma-1}G + \frac{2\nu}{\gamma-1}H + \\ &\qquad + \frac{\nu}{(\gamma-1)^2}\left[\frac{4\gamma-(\gamma-1)^2}{\gamma} + A_1(\gamma+1)\right] = 0 \\ &\lambda F' + \lambda G' + 2(\nu-1)F + \frac{2\nu(\gamma+1)}{(\gamma-1)^2} - \frac{\nu(\gamma+1)}{\gamma-1}G = 0 \\ &\lambda\gamma F' + \lambda H' + \nu(\gamma+1)F - \frac{\nu(\gamma+1)}{\gamma-1}H + \frac{\gamma+1}{\gamma-1}\nu\left(\frac{\gamma-1}{2\gamma} - A_1\right) = 0 \end{aligned} \tag{3.2}$$

This system is a system of equations with constant coefficients, if we take $l = \ln\lambda$ as the independent variable.

For the solution of system (3.2), it would be possible to use integral (2.1) which for $\omega = \omega_1$ has the form:

$$\begin{aligned} &\left[\frac{\nu(2\gamma-1)}{\omega_1-2\nu} + \gamma\right]F + \frac{\nu(2\gamma-1)}{\omega_1-2\nu}G + H = \\ &= \left[\frac{(3\gamma-1)(\gamma+1)}{2\gamma(\gamma-1)} + A_1\right]\lambda^{\frac{\nu(\gamma+1)}{\gamma-1}} + \frac{2\nu}{\gamma-1}\,\frac{2\gamma-1}{\omega_1-2\nu} + \frac{\gamma-1}{2\gamma} - A_1 \end{aligned} \tag{3.3}$$

But because the complete system (3.2) is easily integrated, one need not use the integral (3.3). However, it can be useful in computing the dependence of the required functions on λ and also as a check on the calculation.

We shall find the general solution of the homogeneous system of equations (3.2). The auxiliary equation of the system (3.2) is written thus:

$$\begin{vmatrix} n-\frac{2(\nu+1)}{\gamma-1} & -\frac{2\nu}{\gamma-1} & \frac{2}{\gamma-1}(n+\nu) \\ n+2(\nu-1) & n-\frac{\nu(\gamma+1)}{\gamma-1} & 0 \\ \gamma n+\nu(\gamma+1) & 0 & n-\frac{\nu(\gamma+1)}{\gamma-1} \end{vmatrix} = 0 \qquad (3.4)$$

To each root n_i of equation (3.4) there corresponds a solution $\exp(n_i l) = \lambda^{n_i}$ of the homogeneous system.

The first root of equation (3.4) is equal to

$$n_1 = \nu(\gamma+1)/(\gamma-1) \qquad (3.5)$$

The second and third roots of equation (3.4) satisfy the following quadratic equation:

$$n^2 + \frac{5\nu\gamma+3\nu+2}{\gamma+1}n - \frac{2\nu(\nu\gamma+1)(3-\gamma)}{\gamma^2-1} = 0$$

and are equal to

$$n_2 = -\frac{b_1}{2} + \sqrt{\frac{b_1^2}{4}+b_2}, \qquad n_3 = -\frac{b_1}{2} - \sqrt{\frac{b_1^2}{4}+b_2} \qquad (3.6)$$

where

$$b_1 = \frac{5\nu\gamma+3\nu+2}{\gamma+1}, \qquad b_2 = \frac{2\nu(\nu\gamma+1)(3-\gamma)}{\gamma^2-1}$$

From (3.6) it is seen that $n_2 = 0$ for $\gamma = 3$. The dependence of the roots n_1, n_2 and n_3 on ν and γ are given in the Table.

The general solution of a homogeneous system corresponding to system (3.2) can be represented in the form

$$F(\lambda) = c_2\lambda^{n_2} + c_3\lambda^{n_3}$$

$$G(\lambda) = c_1\lambda^{n_1} + \frac{(n_2+2\nu-2)(\gamma-1)}{(\gamma+1)\nu-(\gamma-1)n_2}c_2\lambda^{n_2} + \frac{(n_3+2\nu-2)(\gamma-1)}{(\gamma+1)\nu-(\gamma-1)n_3}c_3\lambda^{n_3} \qquad (3.7)$$

$$H(\lambda) = \frac{\gamma-1}{2\gamma}c_1\lambda^{n_1} + \frac{(\gamma n_2+\nu\gamma+\nu)(\gamma-1)}{(\gamma+1)\nu-(\gamma-1)n_2}c_2\lambda^{n_2} + \frac{(\gamma n_3+\nu\gamma+\nu)(\gamma-1)}{(\gamma+1)\nu-(\gamma-1)n_3}c_3\lambda^{n_3}$$

The particular solution of the non-homogeneous system of equations (3.2) will thus be

$$F = \alpha_1, \qquad G = \alpha_2, \qquad H = \alpha_3 \qquad (3.8)$$

where a_1, a_2, a_3 are expressed in terms of A_1, ν and γ according to the formulas

$$\alpha_1 = \frac{\nu(\gamma+1)}{2(\gamma-1)(\nu\gamma+1)} A_1, \qquad \alpha_2 = \frac{\nu-1}{\nu\gamma+1} A_1 + \frac{2}{\gamma-1} \tag{3.9}$$
$$\alpha_3 = \frac{1}{2}\left[\frac{\gamma-1}{\gamma} - \frac{\nu\gamma-\nu+2}{\nu\gamma+1} A_1\right]$$

Using (3.7) and (3.8) we find the solution of system (1.13) for $\omega = \omega_1$ in the following form

$$f_1(\lambda) = \frac{1-\gamma}{\gamma+1}\lambda[\alpha_1 + c_2\lambda^{n_2} + c_3\lambda^{n_3}]$$
$$g_1(\lambda) = \lambda^{\nu-2}\left[\alpha_2 + c_1\lambda^{n_1} + \frac{(n_2+2\nu-2)(\gamma-1)}{\nu(\gamma+1)-(\gamma-1)n_2} c_2\lambda^{n_2} + \right.$$
$$\left. + \frac{(n_3+2\nu-2)(\gamma-1)}{\nu(\gamma+1)-(\gamma-1)n_3} c_3\lambda^{n_3}\right] \tag{3.10}$$
$$h_1(\lambda) = \lambda^{\nu}\left[\alpha_3 + \frac{\gamma-1}{2\gamma} c_1\lambda^{n_1} + \frac{(\gamma n_2+\nu\gamma+\nu)(\gamma-1)}{\nu(\gamma+1)-(\gamma-1)n_2} c_2\lambda^{n_2} + \right.$$
$$\left. + \frac{(\gamma n_3+\nu\gamma+\nu)(\gamma-1)}{\nu(\gamma+1)-(\gamma-1)n_3} c_3\lambda^{n_3}\right]$$

We shall determine the constants c , c_2, c_3 and A_1 so that the functions $f_1(\lambda)$, $g_1(\lambda)$, $h_1(\lambda)$ satisfy the boundary conditions (1.14).

Thus, from the latter equation of (1.14) (the velocity at the center must be equal to zero), taking into consideration that n_3 for arbitrary γ is a negative quantity whose modulus is greater than unity, we obtain $c_3 = 0$; from the other equations of (1.14) we obtain a system of non-homogeneous linear equations with coefficients which depend on γ and ν.

$$\alpha_1 + c_2 + \frac{2}{\gamma-1} = 0 \tag{3.11}$$

$$\alpha_2 + c_1 + \frac{(n_2+2\nu-2)(\gamma-1)}{\nu(\gamma+1)-(\gamma-1).n_2} c_2 = 0 \tag{3.12}$$

$$\alpha_3 + \frac{\gamma-1}{2\gamma} c_1 + \frac{(\gamma n_2+\nu\gamma+\nu)(\gamma-1)}{\nu(\gamma+1)-(\gamma-1)n_2} c_2 = 0 \tag{3.13}$$

4. Results of calculations according to formulas of the exact solution. If c_1 from (3.12) is substituted into equation (3.13) and then equations (3.11) and (3.9) are used, we obtain a relation for determining A_1. The quantity A_1 will depend on γ and ν.

After determining A_1 from the system of equations (3.9), (3.11), (3.12), we find the dependence on γ and ν of the quantities a_1, a_2, a_3, c_1, c_2.

The calculation formulas for finding the indicated quantities have the form

$$A_1 = \frac{B_6}{B_5}, \qquad \alpha_1 = \frac{1}{\gamma - 1} B_3 A_1, \qquad \alpha_2 = \frac{\nu - 1}{\nu\gamma + 1} A_1 + \frac{2}{\gamma - 1}$$

$$\alpha_3 = \frac{\gamma - 1}{2\gamma} - B_4 A_1, \qquad c_2 = \frac{2}{\gamma - 1} - \alpha_1, \qquad c_1 = -[\alpha_2 + (\gamma - 1) B_1 c_2]$$

where

$$B_1 = \frac{n_2 + 2\nu - 2}{\nu(\gamma + 1) - (\gamma - 1) n_2}, \qquad B_2 = \frac{\gamma n_2 + \nu\gamma + \nu}{\nu(\gamma + 1) - (\gamma - 1) n_2}$$

$$B_3 = \frac{\nu(\gamma + 1)}{2(\nu\gamma + 1)}, \qquad B_4 = \frac{\nu\gamma - \nu + 2}{2(\nu\gamma + 1)}$$

$$B_5 = \frac{\gamma - 1}{2\gamma}\left(B_1 B_3 - \frac{\nu - 1}{\nu\gamma - 1}\right) - B_2 B_3 - B_4, \qquad B_6 = \frac{3 - \gamma}{2\gamma} + \frac{\gamma - 1}{\gamma} B_1 - 2B_2$$

TABLE 1.

ν	γ	ω_1	n_1	n_2	$-n_3$	α_1	α_2	$-\alpha_3$	$-c_1$	c_2	A_1
	1.2	2.6364	33.000	5.9131	19.095	7.0942	10.860	0.4756	11.923	2.9058	1.9778
	1.4	2.3333	18.000	3.1540	16.487	3.2461	5.7213	0.4342	6.5665	1.7540	1.8755
3	$^5/_3$	2.0000	12.000	1.7684	15.268	1.8211	3.6070	0.4070	4.2717	1.1789	1.8211
	3.0	1.0000	6.000	0.0000	14.000	0.5333	1.3556	0.3778	1.6667	0.4667	1.7778
	7.0	0.0000	4.000	—0.8031	13.697	0.1635	0.4968	0.3887	0.6098	0.1699	1.7979
	1.2	1.8182	22.000	4.1894	13.280	6.6951	10.609	0.6470	13.228	3.3049	2.0694
	1.4	1.6667	12.000	2.2240	11.391	3.1520	5.5253	0.5926	6.3238	1.8481	1.9962
2	$^5/_3$	1.5000	8.000	1.2394	10.489	1.7715	3.4429	0.5381	4.0315	1.2285	1.9191
	3.0	1.0000	4.000	0.0000	9.500	0.5238	1.2619	0.4524	1.5000	0.4762	1.8333
	7.0	0.5000	2.667	—0.5431	9.207	0.1621	0.4549	0.4225	0.5326	0.1712	1.8237

The results of the calculations are presented in the table.

The formulas (3.10) taking into account that $c_3 = 0$ are written thus:

$$f_1(\lambda) = \frac{1 - \gamma}{\gamma + 1} \lambda (\alpha_1 + c_2 \lambda^{n_2})$$

$$g_1(\lambda) = \lambda^{\nu - 2}[\alpha_2 + c_1 \lambda^{n_1} + (\gamma - 1) B_1 c_2 \lambda^{n_2}] \tag{4.1}$$

$$h_1(\lambda) = \lambda^{\nu}\left[\alpha_3 + \frac{\gamma - 1}{2\gamma} c_1 \lambda^{n_1} + (\gamma - 1) B_2 c_2 \lambda^{n_2}\right]$$

From the relation (0.3) in the case of spherical waves ($\nu = 3$), cylindrical waves ($\nu = 2$) and plane waves ($\nu = 1$) we will have respectively

$$\omega_1 = \frac{7 - \gamma}{\gamma + 1}, \qquad \omega_1 = \frac{4}{\gamma + 1}, \qquad \omega_1 = 1$$

The formula for finding the mass of matter enclosed in some finite region containing the origin of the coordinate system

$$m = \sigma_\nu \int_0^r \rho r^{\nu-1}\, dr$$

shows that the initial mass will be of finite magnitude only in the spherical and cylindrical cases. Therefore, the calculations were carried out for these two cases.

The integral (3.3) in the case of spherical symmetry takes the form

$$\frac{(\gamma-3)(\gamma-1)}{7\gamma-1}F - \frac{3(2\gamma-1)(\gamma+1)}{7\gamma-1}G + H =$$

$$= \left[\frac{(3\gamma-1)(\gamma+1)}{2\gamma(\gamma-1)} + A_1\right]\lambda^{\frac{3(\gamma+1)}{\gamma-1}} - \frac{6}{\gamma-1}\,\frac{(2\gamma-1)(\gamma+1)}{7\gamma-1} + \frac{\gamma-1}{2\gamma} - A_1$$

and in the cylindrical case

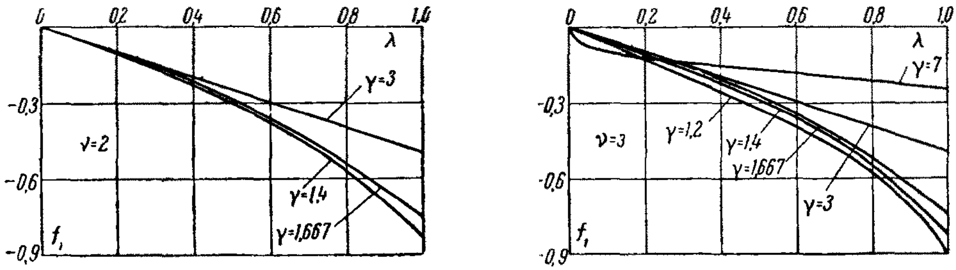
$$-\frac{\gamma-1}{2\gamma}F - \frac{(2\gamma-1)(\gamma+1)}{2\gamma}G + H = \left[\frac{(3\gamma-1)(\gamma+1)}{2\gamma(\gamma-1)} + A_1\right]\lambda^{\frac{2(\gamma+1)}{\gamma-1}} +$$

$$+ \frac{\gamma-1}{2\gamma} - \frac{(2\gamma-1)(\gamma+1)}{\gamma(\gamma-1)} - A_1$$

The dependence of $R_2(q)$ and $r(q)$ is found from the relations (2.4) and (2.5) in which it is necessary to set $\delta = \delta_1$, where

$$\delta_1 = \delta(\omega_1) = \frac{\gamma+1}{\nu\gamma-\nu+2}$$

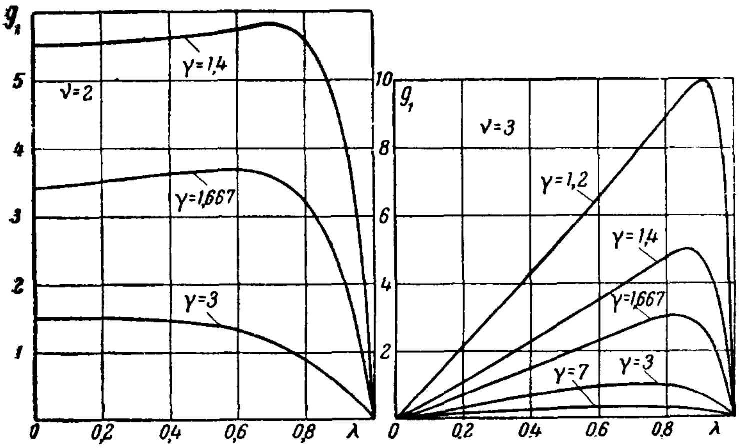
Using the calculated constants and formulas (4.1), the dependence of $f_1(\lambda)$, $g_1(\lambda)$ and $h_1(\lambda)$ (Figs. 1,2,3 respectively) were constructed for various values of γ and ν. With the help of the functions $f_1(\lambda)$, $g_1(\lambda)$ and $h_1(\lambda)$, and knowing the constant A_1, it is possible to calculate the characteristics of the motion for small values of q.



a *b*

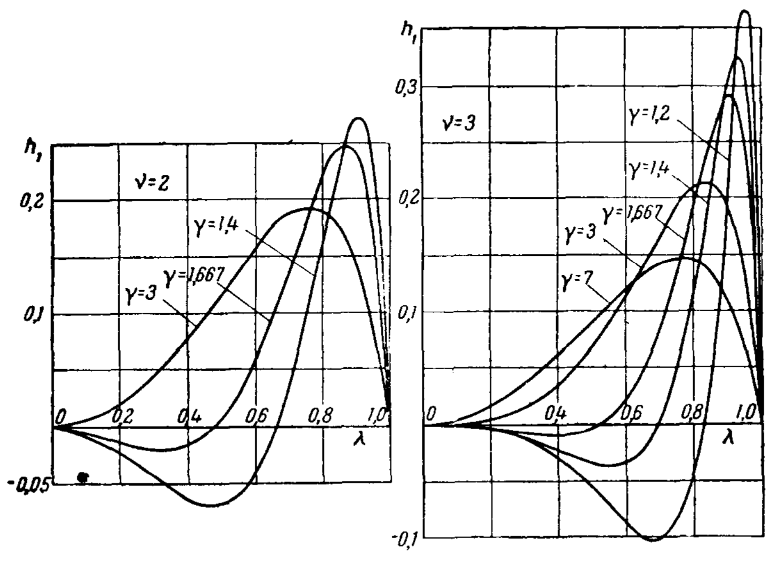
Fig. 1.

In Figs. 4*a* and 4*b* the distribution of the non-dimensional pressure $h = h_0 + qh_1$ is given in the spherical case for $\gamma = 1.4$ and $\gamma = 7$ and in the cylindrical case for $\gamma = 1.4$ for the values $q = 0$ and $q = 0.2$.



a б

Fig. 2.

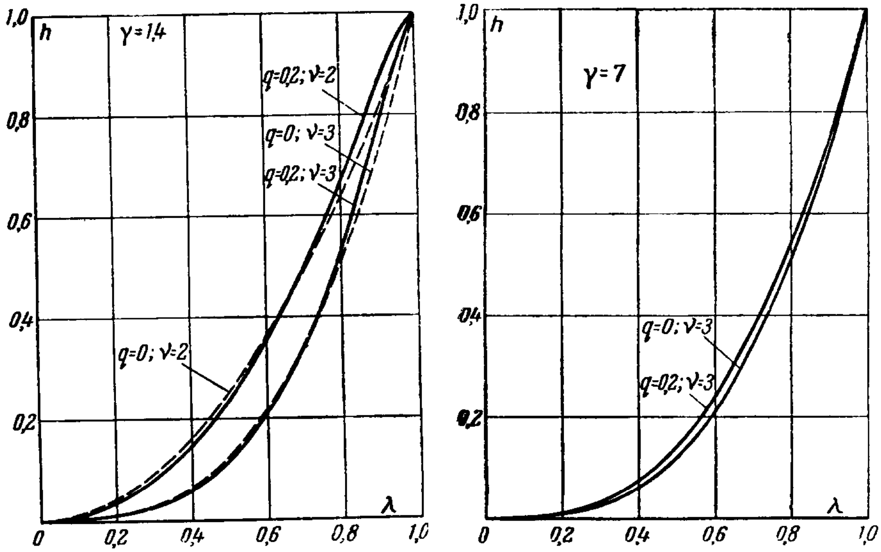


a б

Fig. 3.

These graphs show the influence of back pressure on the development of an explosion in the initial stage.

If the values for ω_1 found for $\nu = 3$ and $\nu = 2$ are equated, then we obtain $\gamma = 3$. In this case $\omega_1 = 1$. A comparison of the values of the non-dimensional pressure for $\gamma = 3$, $\omega_1 = 1$ and various $\nu (\nu = 3,\ \nu = 2)$ is given in Fig. 5.



a *b*

Fig. 4.

We note in conclusion that the exact solution for $\omega = \omega_1$ in the particular case $\nu = 3$ obtained in the present paper supplements and makes more precise the results of [7] (see also [8]).

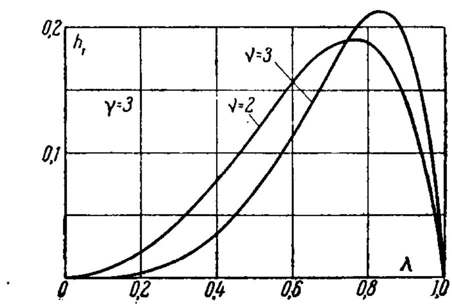


Fig. 5.

The author sincerely thanks N.S. Mel'nikov for discussing certain questions which are examined in the present paper.

BIBLIOGRAPHY

1. Sedov, L.I., *Metody podobiia i razmernosti v mekhanike (Methods of Similarity and Dimensional Analysis in Mechanics)*. 4th Ed. Moscow, 1957.

2. Korobeinikov, V.P. and Riazanov, E.V., Predstavlenie resheniia zadachi o tochechnom vzryve v gaze v osobykh sluchaiakh (Presentation of a solution of the problem of a point explosion in a gas in special cases). *PMM* Vol. 23, No. 2, 1959.

3. Okhotsimskii, D.E., Kondrasheva, I.L., Vlasova, Z.P. and Kazakova, R.K., Raschet tochecknogo vzryva s uchetom protivodavleniia (Calculation of a point explosion including back pressure). *Tr. Matem. in-ta Akad. Nauk SSSR*, Vol. 50, 1957.

4. Sakurai, A., On propagation and structure of the blast wave. *J. Phys. Soc. Japan*, Vol. 8, No. 5, 1953; Vol. 9, No. 2, 1954.

5. Brushlinskii, D.N. and Solomakhova, T.S., Issledovanie zadachi o cil'nom vzryve s uchetom protivodavleniia (Investigation of the problem of a strong explosion including back pressure). Sb. st. No. 19, No. 7, *Theoretical Hydrodynamics*, edited by L.I. Sedov, Oborongiz, 1956.

6. Lidov, M.L., K teorii linearizirovannykh reshenii okolo odnomernykh avtomodel'nykh dvizhenii gaza (On the theory of the linearized solution of the one-dimensional self-modeling motion of a gas). *Dokl. Akad. Nauk SSSR*, Vol. 102, No. 6, 1955.

7. Korobeinikov, V.P. and Mel'nikova, N.S., O tochnykh resheniiakh linearizirovannoi zadachi o tochechnom vzryve s protivodavleniem (On exact solutions of a linearized problem of a point explosion with back pressure). *Dokl. Akad. Nauk SSSR*, Vol. 116, No. 2, 1957.

8. Korobeinikov, V.P. and Mel'nikova, N.S., Pis'mo v redaktsiiu (Letter to the editor). *Dokl. Akad. Nauk SSSR*, Vol. 126, No. 1, 1959.

Translated by R.D.C.